# Light Scattering Group

## Light Scattering Inversion

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A method of inverting the Mie light scattering equation of spherical homogeneous particles of real and complex argument is being investigated. The aims are to obtain a mathematical proof which shows it possible to uniquely determine the size parameters of a particle from its light scattering irradiance function and to develop mathematical methods and computational procedures for this to be achieved in practice.

Research by us [1,2] has shown that unique values of $\alpha$ and $\beta$ can be found from the Mie coefficients $a_n$ and $b_n$ or the amplitude functions $S_\perp(z)$ and $S_\parallel(z)$ for real and complex particle parameters. This can be shown by starting with the electromagnetic field $(E,H)$ in a linear, isotropic, homogeneous medium which satifies the homogeneous vector Helmholtz equations [3]

$$\nabla^2 \overline{E} + k^2 \overline{E} = 0 \quad 4.0$$

$$\nabla^2 \overline{H} + k^2 \overline{H} = 0 \quad 4.1$$

where $k^2 = \omega^2 \varepsilon \mu$. To solve the homogeneous vector Helmholtz equation in spherical polar coordinates we first construct the two divergence free vector functions

$$\overline{M} = \nabla \times \left(\overline{r}\Psi\right) \quad 4.2$$

$$\overline{N} = \frac{\nabla \times \overline{M}}{k} \quad 4.3$$

where $\overline{r}$ is an arbitrary vector and $\Psi$ a given scalar. The problem has now been reduced to finding the solution of the scalar equation in spherical polar coordinates because $\overline{M}$ is a solution of the vector equation $\nabla^2 \overline{M} + k^2 \overline{M} = 0$ if $\Psi$ is a solution of the scalar Helmholtz equation.

Solutions of the scalar Helmholtz equation in spherical polar coordinates are given by

$$\psi(r,\vartheta,\phi) = \Phi(\phi)\Theta(\vartheta)R(r) \quad 4.4$$

where the three functions are respective solutions of the equations

$$\frac{d^2\Phi}{d\phi^2} + m^2\phi = 0 \quad 4.5a$$

$$\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta\frac{d\Theta}{d\vartheta}\right)+\left\{n(n+1)-\frac{m^2}{\sin^2\vartheta}\right\}\Theta=0 \qquad 4.5b$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left\{k^2r^2-n(n+1)\right\}R=0 \qquad 4.5c$$

.

Solving the above equations gives two linearly independent solutions

$$\psi_{emn}=\cos(m\phi)P_n^m(\cos\vartheta)z_n(kr) \qquad 4.6a$$

.

and

$$\psi_{omn}=\sin(m\phi)P_n^m(\cos\vartheta)z_n(kr) \qquad 4.6b$$

,

where $z_n$ is one of the four spherical Bessel functions $j_n$, $n_n$, $h_n^1=j_n+in_n$ or $h_n^2=j_n-in_n$ and $P_n^m(\cos\vartheta)$ is the associated Legendre function of order $n$, degree $m$ ( $0\le m\le n$ ). Using these results we obtain explicit expressions for the vector spherical harmonics

$$\overline{M}_{pmn}^{1,3}=\nabla\times\left(\bar{r}\psi_{pmn}^{1,3}\right);\quad \overline{N}_{pmn}^{1,3}=\frac{\nabla\times\overline{M}_{pmn}^{1,3}}{k}, \qquad 4.7$$

where the superscripts 1 and 3 denote $j_n(kr)$ and $h_n^1(kr)$ respectively and $p$ specifies either even or odd functions. Linear combinations of these vector spherical harmonics will give the magnetic and electric field vectors of the incident, scattered and internal fields. Those of the scattered field are

$$\overline{E}^s(\bar{r})=E_0\sum_{n=1}^{\infty}i^n\frac{(2n+1)}{n(n+1)}\left\{ia_n\overline{N}_{e1n}^3(kr)-b_n\overline{M}_{o1n}^3(kr)\right\} \qquad 4.8a$$

$$\overline{H}^s(\bar{r})=\frac{k}{\omega\mu_0}E_0\sum_{n=1}^{\infty}i^n\frac{(2n+1)}{n(n+1)}\left\{ib_n\overline{N}_{o1n}^3(kr)+a_n\overline{M}_{e1n}^3(kr)\right\} \qquad 4.8b$$

,

where $a_n$ and $b_n$ are the scattering coefficients which can be obtained by using the continuity of the transverse field components at the boundary of the spherical particle. Similar coefficients may also be obtained for the internal fields using the same principle.

With results collected in the far field region the vector spherical harmonics simplify to give

$$N_{1n}^3(\bar{r})\to i\frac{e^{ikr}}{i^{n+1}kr}\overline{B}_{1n}(\vartheta,\phi) \qquad 4.9a$$

$$M_{1n}^{3}(\bar{r}) \rightarrow \frac{e^{ikr}}{i^{n+1}kr}\bar{C}_{1n}(\vartheta,\phi) \qquad 4.9b$$
,

where

$$\bar{B}_{1n}(\vartheta,\phi) = \left[\hat{a}_{\vartheta}\tau_{n}(\vartheta) + i\hat{a}_{\phi}\pi_{n}(\vartheta)\right]e^{i\phi} \qquad 4.10a$$

$$\bar{C}_{1n}(\vartheta,\phi) = \left[i\hat{a}_{\vartheta}\pi_{n}(\vartheta) - \hat{a}_{\phi}\tau_{n}(\vartheta)\right]e^{i\phi} \qquad 4.10b$$

are angular eigenvectors of the vector wave equation in spherical polar coordinates.

The scattered field from a sphere can then generally be written as a linear combination of the above eigenvectors to yield

$$\bar{E}^{s}(\bar{r}) = \frac{e^{ikr}}{r}E_{0}\bar{F}(\vartheta,\phi) \qquad 4.11$$
,

where

$$\bar{F}(\vartheta,\phi) = \frac{i}{2k}\left[\left(p_{x} - ip_{y}\right)\sum_{n=1}\frac{2n+1}{n(n+1)}\left(a_{n}\bar{B}_{1n} - ib_{n}\bar{C}_{1n}\right) + \left(p_{x} + ip_{y}\right)\sum_{n=1}\frac{2n+1}{n(n+1)}\left(a_{n}\bar{B}_{-1n} + ib_{n}\bar{C}_{-1n}\right)\right] \qquad 4.12$$
,

for an incident electric field having unit polarisation vector $\bar{p} = \hat{a}_{x}p_{x} + \hat{a}_{y}p_{y}$ and amplitude $E_{0}$. These orthogonal functions may be applied to obtain the multipole coefficients $a_{n}$ and $b_{n}$. By utilising the orthogonality of the angular vector functions we obtain

$$\frac{1}{\Lambda_{n}}\int \bar{F}^{R}(\vartheta,\phi)\cdot\bar{B}_{1n}d\Omega = \frac{i}{k\sqrt{2}}\frac{1}{\Lambda_{n}}\int\left(S_{\parallel}\tau_{n} + S_{\perp}\pi_{n}\right)d\Omega = \frac{i}{k\sqrt{2}}\frac{2n}{n(n} \qquad 4.13$$
.

Hence, if $S_{\perp}(\vartheta)$ and $S_{\parallel}(\vartheta)$ are known functions, the scattering coefficients can be found by

$$a_{n} = \frac{1}{2n(n+1)}\int_{0}^{\pi}\left[S_{\parallel}(\vartheta)\tau_{n}(\vartheta) + S_{\perp}(\vartheta)\pi_{n}(\vartheta)\right]\sin\vartheta d\vartheta \qquad 4.14a$$
.

$$b_{n} = \frac{1}{2n(n+1)}\int_{0}^{\pi}\left[S_{\parallel}(\vartheta)\pi_{n}(\vartheta) + S_{\perp}(\vartheta)\tau_{n}(\vartheta)\right]\sin\vartheta d\vartheta \qquad 4.14b$$
.

these coefficients allow unique values of $\alpha$ and $\beta$ to be determined. Associated with a particular spherical particle there exists a particular set of boundary conditions which are satisfied for all values of the order $n \geq 1$ for which $a_n$ and $b_n$ contribute to the scattering pattern. We note that these equations are dependent only on the relative refractive index and radius of the particle together with a constant - the propagation constant in the ambient medium. The set of equations are therefore unique to the particle.

The internal Mie coefficients $c_n$, $d_n$ and the internal Riccati-Bessel functions $\psi_n(\beta)$, $\psi_n'(\beta)$ can however be eliminated from the equations to give

$$m^2 = \left[\frac{a_n\zeta_n(\alpha)-\psi_n(\alpha)}{a_n\zeta_n'(\alpha)-\psi_n'(\alpha)}\right]\left[\frac{b_n\zeta_n'(\alpha)-\psi_n'(\alpha)}{b_n\zeta_n(\alpha)-\psi_n(\alpha)}\right]. \quad 4.15$$

Hence for a known set of Mie coefficients $a_n$ and $b_n$, the particle parameters may be extracted by plotting the function

$$F_n(z) = \left[\frac{a_n\zeta_n(z)-\psi_n(z)}{a_n\zeta_n'(z)-\psi_n'(z)}\right]\left[\frac{b_n\zeta_n'(z)-\psi_n'(z)}{b_n\zeta_n(z)-\psi_n(z)}\right] \quad 4.16$$

against order $n$ for different values of $z$ until a straight line parallel to the order axis is obtained. When this condition is satisfied $z = \alpha$ and $F_n(z) = m^2$ as is demonstrated in Fig. (4.1). Furthermore, since the Riccati-Bessel functions are non-linear in $z$, no other value of $z$ will yield the required straight line. Thus unique values of $\alpha$ and $\beta$ can be found from the Mie coefficients or the amplitude functions $S_{\parallel}(\vartheta)$ and $S_{\perp}(\vartheta)$ because of Eqs. (4.14).

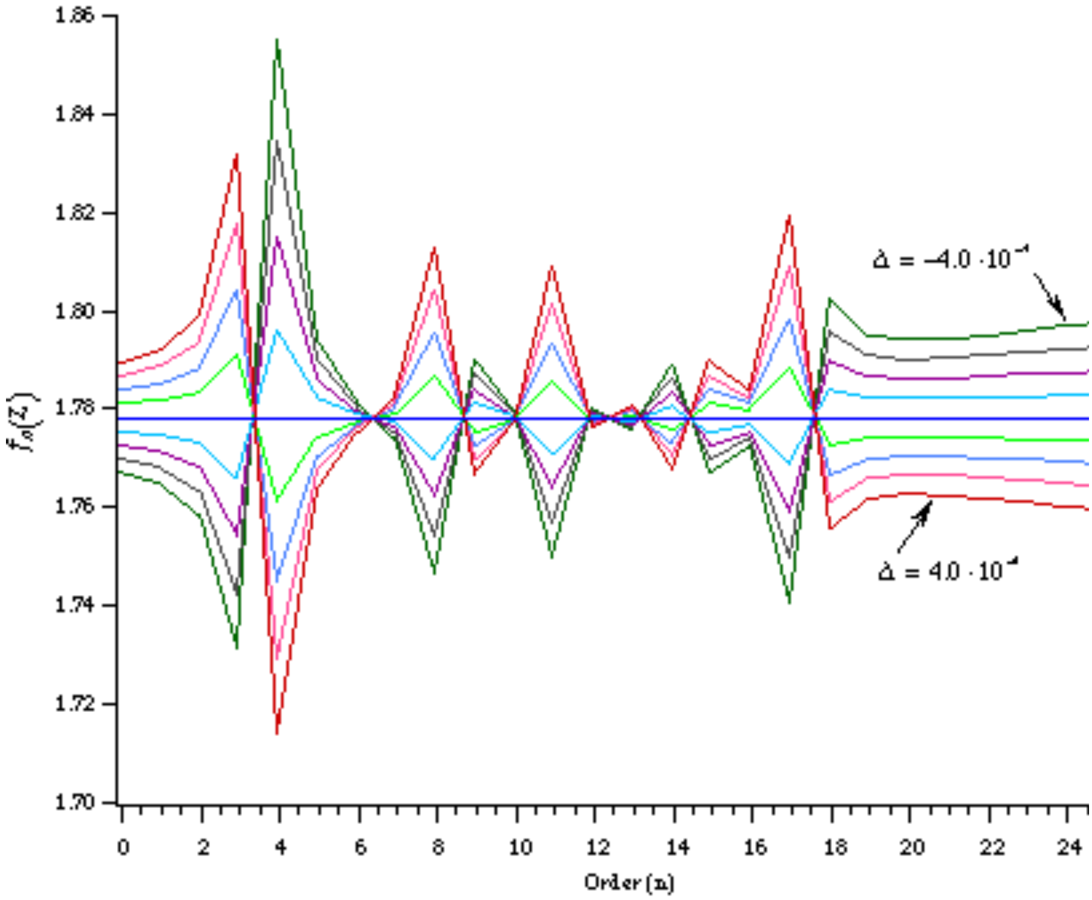


Fig 4.1

Inversion plot for $\alpha = 15$, $\beta = 20$. The fractional deviation $\Delta = (z/\alpha) - 1$ is incremented in steps of $10^{-4}$ between $-4.0 \times 10^{-4}$ and $4.0 \times 10^{-4}$.

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References


1. Everitt, J. (1999). Gegenbauer Analysis of Light Scattering from Spheres. Physics. Hatfield, University of Hertfordshire, England
2. Ludlow, I. K. and J. Everitt (2000). "Inverse Mie problem." Physics 17(12): 2229-2235.
3. Stratton, J. A. (1941). The Field Equations; The Hertz Vectors, or Polarisation Potentials. Electromagnetic theory. F. K. Richtmyer. New York, McGraw-Hill Book Company, Inc.: 28-32.