Discrete Gegenbauer Analysis

The conventional way of representing the light scattering of spheres consists of a graph of $I_\wp(\vartheta) = \left|S_\wp(z)\right|^2$ against the scattering angle ϑ in the range $0^\circ \le \vartheta \le 180^\circ$, this is the ϑ-plot. However we may express the scattering irradiance as the discrete Gegenbauer series [1]

$$I_\wp(\vartheta) = \left|S_\wp(z)\right|^2 = \sum_{l=0}^{\infty}\left(C_l^1\right)_\wp T_l^1(z) \qquad 2.0$$

,

where $-1 \le z = \cos\vartheta \le 1$, $T_n^1(z)$ are Gegenbauer functions and the value of $k = 1$ has been chosen as it defines the most relavant solutions.

Examples of the z-plots and their associated Gegenbauer spectra are shown in Figs. (2.1) for $\left|S_\perp(z)\right|^2$ and $\left|S_\parallel(z)\right|^2$ of a sphere having $\alpha = 25$ and $\beta = 30$. The envelopes of the Gegenbauer spectra are clearly simpler than the original scattering patterns and the number of coefficients present is generally smaller than the number of points required to construct the z-plot. Such Spectra are in all respects equivalent to ether ϑ-plots or z-plots..

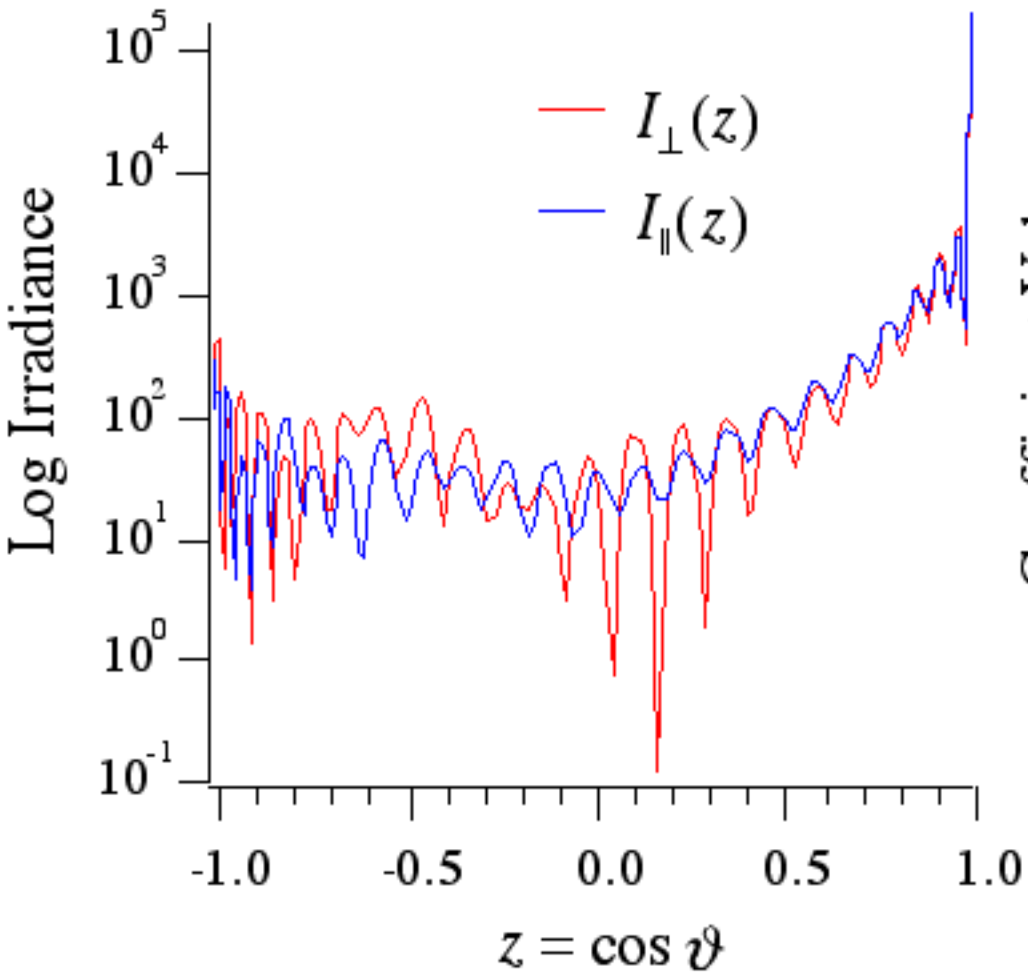

Figure 2.1a

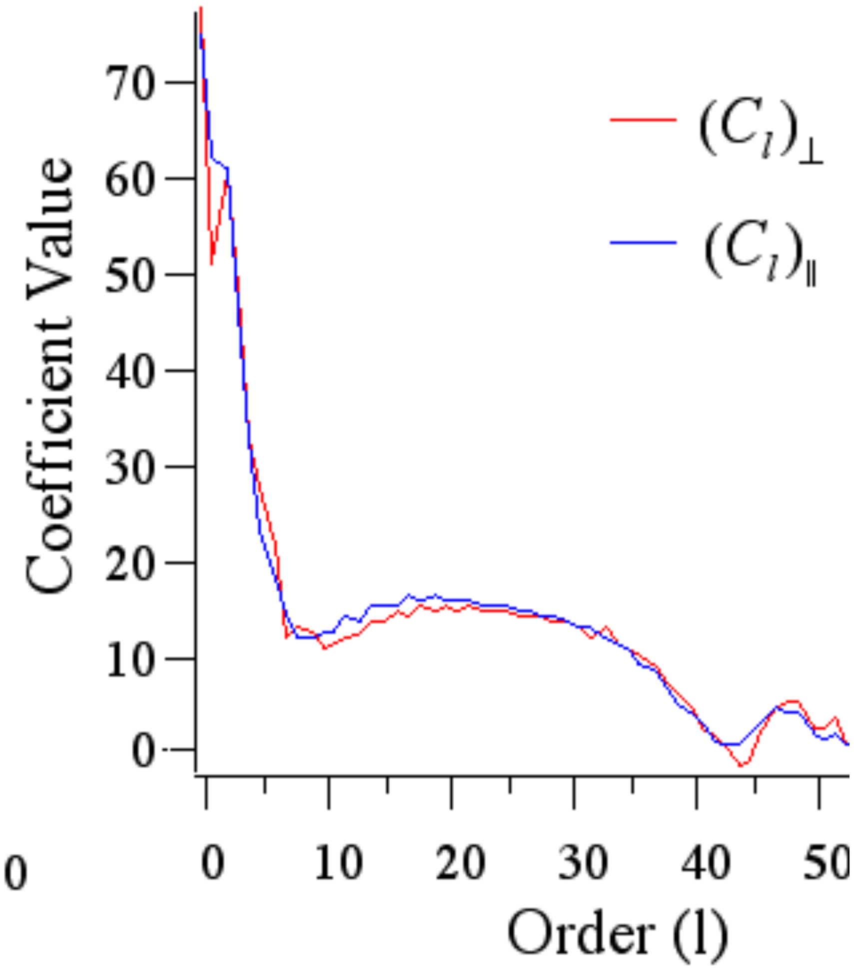

Figure 2.1b

Angular light scattering irradiance function of a particle $\alpha = 25$, $\beta = 30$ as a function of $z = \cos\vartheta$ and the first degree Gegenbauer spectra.

The main advantages of representing the angular scattering functions as discrete Gegenbauer series are:

1. A high cutoff l_{CO} exists above which the coefficients $\left(C_l\right)_{\wp}$ are generally negligible. This cut off gives an estimate of α since $l_{CO} \approx 2\alpha$,
2. They allow more efficient matching of experimental and theoretical patterns than for angular patterns,
3. By treating a spectrum as a distribution, a set of moments can be calculated which characterise the spectrum and
4. The elimination of the angular variable from the scattering pattern gives Gegenbauer coefficients $\left(C_l\right)_{\wp}$ which depend only on α, β and the order *n*. Thus the Gegenbauer transform of the irradiance function can be regarded as a partial inversion of the scattering.

Mie's equations have been reduced to the form

$$S_{\wp}(z) = \sum_{n=0}^{\infty} \left(c_n^1\right)_{\wp} T_n^1(z), \qquad 2.1$$

where the first degree Gegenbauer amplitude coefficients are related to the Mie multipole coefficients using

$$\left(c_n^1\right)_{\perp} = \frac{n}{n+1} b_n + \frac{2n+3}{(n+1)(n+2)} a_{n+1} - \frac{n+3}{n+2} b_{n+2}, \qquad 2.2$$

and

$$\left(c_n^1\right)_{\parallel} = \frac{n}{n+1} a_n + \frac{2n+3}{(n+1)(n+2)} b_{n+1} - \frac{n+3}{n+2} a_{n+2}. \qquad 2.3$$

We take the irradiance function of the scattering pattern to be

$$I_{\wp}(z) = \left|S_{\wp}(z)\right|^2 = \left|\sum_{n=0}^{\infty}\left(c_n^1\right)_{\wp} T_n^1(z)\right|^2 . \quad 2.4$$

Thus the Gegenbauer irradiance coefficients $\left(C_l^1\right)_{\wp}$ are related to the Gegenbauer amplitude coefficients $\left(c_n^1\right)_{\wp}$ by

$$\sum_{l=0}^{\infty}\left(C_l^1\right)_{\wp} T_l^1(z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(c_n^1\right)_{\wp}\left(c_m^1\right)_{\wp}^{*} T_n^1(z) T_m^1(z) . \quad 2.5$$

An expansion of the product of two Gegenbauer polynomials as a Gegenbauer series is given as

$$T_m^1(z) T_n^1(z) = \sum_{l=|m-n|_{(2)}}^{m+n} {}^1W_l^{n,m}\, T_l^1(z) , \quad 2.6$$

in which $l = m+n,\, m+n-2, \ldots |m-n|$ and the product coefficient is given by

$${}^1W_l^{n,m} = \frac{1}{4}\frac{(2l+3)}{(l+1)(l+2)}\frac{(s+1)!(s+2)!}{(2s+3)!}\frac{(2a+2)!}{a!(a+1)!}\frac{(2b+2)!}{b!(b+1)!}\frac{(2c+2)!}{c!(c+1)!} . \quad 2.7$$

for $s = (l+m+n)/2$, $a = (m+n-l)/2$, $b = (l+m-n)/2$ and $c = (l+n-m)/2$.

In the standard experimental arrangement, scattering patterns are recorded in a fixed plane of detection, $\phi = 90^\circ$, and the only variable is the scattering angle ϑ in the range $0^\circ \leq \vartheta \leq 180^\circ$. The scattered field of interest is now represented by eigenvectors which are a subset of the complete scattering. Within this restricted set the eigenvectors are no longer orthogonal in ϑ but a new set of eigenvectors can be defined that are orthogonal. The conversion to the new set is carried out by reformulating the Mie amplitude and irradiance functions as first degree Gegenbauer series. The advantages of the new equations are:

1. Scattering can be represented by Gegenbauer spectra which are independent of the scattering angle ϑ.
2. Such spectra require fewer points and are simpler than the usual scattering patterns.
3. The Gegenbauer series are unique to the particle. They terminate rapidly at high order to give an immediate estimate of α.
4. Approximate scattering formulas can be expanded as Gegenbauer series to allow comparison with rigorous theory
5. The relations between the irradiance and amplitude Gegenbauer coefficients offers a potential means for the inversion of experimental scattering patterns.

References

1. Everitt, J. (1999). Thesis, Gegenbauer Analysis of Light Scattering from Spheres. Theoretical, University of Hertfordshire, England